# FP2: De Moivre's theorem

Past Paper Questions
2006 - 2014

Name:

**Complex numbers**

$$\{r(\cos\theta + \mathrm{i}\sin\theta)\}^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

The roots of $z^n = 1$ are given by $z = e^{\frac{2\pi k \mathrm{i}}{n}}$, for $k = 0, 1, 2, \ldots, n-1$

January 2006

**6** It is given that $z = e^{i\theta}$.

(a) (i) Show that

$$z + \frac{1}{z} = 2\cos\theta$$ *(2 marks)*

(ii) Find a similar expression for

$$z^2 + \frac{1}{z^2}$$ *(2 marks)*

(iii) Hence show that

$$z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$$ *(3 marks)*

(b) Hence solve the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving the roots in the form $a + ib$. *(5 marks)*

June 2006

7 (a) Find the six roots of the equation $z^6 = 1$, giving your answers in the form $e^{i\phi}$, where $-\pi < \phi \leqslant \pi$. *(3 marks)*

(b) It is given that $w = e^{i\theta}$, where $\theta \neq n\pi$.

(i) Show that $\dfrac{w^2 - 1}{w} = 2i\sin\theta$. *(2 marks)*

(ii) Show that $\dfrac{w}{w^2 - 1} = -\dfrac{i}{2\sin\theta}$. *(2 marks)*

(iii) Show that $\dfrac{2i}{w^2 - 1} = \cot\theta - i$. *(3 marks)*

(iv) Given that $z = \cot\theta - i$, show that $z + 2i = zw^2$. *(2 marks)*

(c) (i) Explain why the equation

$$(z + 2i)^6 = z^6$$

has five roots. *(1 mark)*

(ii) Find the five roots of the equation

$$(z + 2i)^6 = z^6$$

giving your answers in the form $a + ib$. *(4 marks)*

February 2007

5 (a) Prove by induction that, if $n$ is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ *(5 marks)*

(b) Find the value of $\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)^6$. *(2 marks)*

(c) Show that

$$(\cos\theta + i\sin\theta)(1 + \cos\theta - i\sin\theta) = 1 + \cos\theta + i\sin\theta$$ *(3 marks)*

(d) Hence show that

$$\left(1 + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6 = 0$$ *(4 marks)*

**6** (a) Find the three roots of $z^3 = 1$, giving the non-real roots in the form $e^{i\theta}$, where $-\pi < \theta \leqslant \pi$. *(2 marks)*

(b) Given that $\omega$ is one of the non-real roots of $z^3 = 1$, show that

$$1 + \omega + \omega^2 = 0$$ *(2 marks)*

(c) By using the result in part (b), or otherwise, show that:

(i) $\dfrac{\omega}{\omega + 1} = -\dfrac{1}{\omega}$; *(2 marks)*

(ii) $\dfrac{\omega^2}{\omega^2 + 1} = -\omega$; *(1 mark)*

(iii) $\left(\dfrac{\omega}{\omega + 1}\right)^k + \left(\dfrac{\omega^2}{\omega^2 + 1}\right)^k = (-1)^k \, 2\cos\frac{2}{3}k\pi$, where $k$ is an integer. *(5 marks)*

June 2007

**3** Use De Moivre's Theorem to find the smallest positive angle $\theta$ for which

$$(\cos\theta + i\sin\theta)^{15} = -i$$ *(5 marks)*

**8** (a) (i) Given that $z^6 - 4z^3 + 8 = 0$, show that $z^3 = 2 \pm 2i$. *(2 marks)*

(ii) Hence solve the equation

$$z^6 - 4z^3 + 8 = 0$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(6 marks)*

(b) Show that, for any real values of $k$ and $\theta$,

$$(z - ke^{i\theta})(z - ke^{-i\theta}) = z^2 - 2kz\cos\theta + k^2$$ *(2 marks)*

(c) Express $z^6 - 4z^3 + 8$ as the product of three quadratic factors with real coefficients. *(3 marks)*

January 2008

1 (a) Express $4+4\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(3 marks)*

(b) Solve the equation

$$z^5 = 4+4\mathrm{i}$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(5 marks)*

6 (a) (i) By applying De Moivre's theorem to $(\cos\theta+\mathrm{i}\sin\theta)^3$, show that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$ *(3 marks)*

(ii) Find a similar expression for $\sin 3\theta$. *(1 mark)*

(iii) Deduce that

$$\tan 3\theta = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$$ *(3 marks)*

(b) (i) Hence show that $\tan\frac{\pi}{12}$ is a root of the cubic equation

$$x^3 - 3x^2 - 3x + 1 = 0$$ *(3 marks)*

(ii) Find two other values of $\theta$, where $0<\theta<\pi$, for which $\tan\theta$ is a root of this cubic equation. *(2 marks)*

(c) Hence show that

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4$$ *(2 marks)*

June 2008

**8** (a) (i) Expand

$$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right) \qquad \textit{(1 mark)}$$

(ii) Hence, or otherwise, expand

$$\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 \qquad \textit{(3 marks)}$$

(b) (i) Use De Moivre's theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$ then

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \textit{(3 marks)}$$

(ii) Write down a corresponding result for $z^n - \dfrac{1}{z^n}$. *(1 mark)*

(c) Hence express $\cos^4\theta\sin^2\theta$ in the form

$$A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$$

where $A$, $B$, $C$ and $D$ are rational numbers. *(4 marks)*

(d) Find $\displaystyle\int \cos^4\theta\sin^2\theta \, \mathrm{d}\theta$. *(2 marks)*

January 2009

**8** (a) Show that

$$(z^4 - \mathrm{e}^{\mathrm{i}\theta})(z^4 - \mathrm{e}^{-\mathrm{i}\theta}) = z^8 - 2z^4\cos\theta + 1 \qquad \textit{(2 marks)}$$

(b) Hence solve the equation

$$z^8 - z^4 + 1 = 0$$

giving your answers in the form $\mathrm{e}^{\mathrm{i}\phi}$, where $-\pi < \phi \leqslant \pi$. *(6 marks)*

(c) Indicate the roots on an Argand diagram. *(3 marks)*

June 2009

**1** Given that $z = 2e^{\frac{\pi i}{12}}$ satisfies the equation

$$z^4 = a(1 + \sqrt{3}\,i)$$

where $a$ is real:

(a) find the value of $a$; *(3 marks)*

(b) find the other three roots of this equation, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

January 2010

**8** (a) (i) Show that $\omega = e^{\frac{2\pi i}{7}}$ is a root of the equation $z^7 = 1$. *(1 mark)*

(ii) Write down the five other non-real roots in terms of $\omega$. *(2 marks)*

(b) Show that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$$ *(2 marks)*

(c) Show that:

(i) $\omega^2 + \omega^5 = 2\cos\dfrac{4\pi}{7}$; *(3 marks)*

(ii) $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7} = -\dfrac{1}{2}$. *(4 marks)*

June 2010

**7 (a) (i)** Express each of the numbers $1 + \sqrt{3}\,i$ and $1 - i$ in the form $re^{i\theta}$, where $r > 0$. *(3 marks)*

**(ii)** Hence express

$$(1 + \sqrt{3}\,i)^8(1 - i)^5$$

in the form $re^{i\theta}$, where $r > 0$. *(3 marks)*

**(b)** Solve the equation

$$z^3 = (1 + \sqrt{3}\,i)^8(1 - i)^5$$

giving your answers in the form $a\sqrt{2}\,e^{i\theta}$, where $a$ is a positive integer and $-\pi < \theta \leqslant \pi$. *(4 marks)*

**8 (a)** Express in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$:

**(i)** $4(1 + i\sqrt{3})$;

**(ii)** $4(1 - i\sqrt{3})$. *(3 marks)*

**(b)** The complex number $z$ satisfies the equation

$$(z^3 - 4)^2 = -48$$

Show that $z^3 = 4 \pm 4\sqrt{3}\,i$. *(2 marks)*

**(c) (i)** Solve the equation

$$(z^3 - 4)^2 = -48$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

**(ii)** Illustrate the roots on an Argand diagram. *(3 marks)*

**(d) (i)** Explain why the sum of the roots of the equation

$$(z^3 - 4)^2 = -48$$

is zero. *(1 mark)*

**(ii)** Deduce that $\cos\frac{\pi}{9} + \cos\frac{3\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = \frac{1}{2}$. *(3 marks)*

June 2011

**7 (a) (i)** Use de Moivre's Theorem to show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

and find a similar expression for $\sin 5\theta$. *(5 marks)*

**(ii)** Deduce that

$$\tan 5\theta = \frac{\tan\theta(5 - 10\tan^2\theta + \tan^4\theta)}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ *(3 marks)*

**(b)** Explain why $t = \tan\frac{\pi}{5}$ is a root of the equation

$$t^4 - 10t^2 + 5 = 0$$

and write down the three other roots of this equation in trigonometrical form.

*(3 marks)*

**(c)** Deduce that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$$ *(5 marks)*

January 2012

**5** Find the smallest positive integer values of $p$ and $q$ for which

$$\frac{\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right)^p}{\left(\cos\frac{\pi}{12} - \mathrm{i}\sin\frac{\pi}{12}\right)^q} = \mathrm{i}$$

*(7 marks)*

**8 (a)** Write down the five roots of the equation $z^5 = 1$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(1 mark)*

**(b)** Hence find the four linear factors of

$$z^4 + z^3 + z^2 + z + 1$$

*(3 marks)*

**(c)** Deduce that

$$z^2 + z + 1 + z^{-1} + z^{-2} = \left(z - 2\cos\frac{2\pi}{5} + z^{-1}\right)\left(z - 2\cos\frac{4\pi}{5} + z^{-1}\right)$$

*(4 marks)*

**(d)** Use the substitution $z + z^{-1} = w$ to show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$. *(6 marks)*

June 2012

**8 (a)** Use De Moivre's Theorem to show that, if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

*(3 marks)*

**(b) (i)** Expand $\left(z^2 + \frac{1}{z^2}\right)^4$. *(1 mark)*

**(ii)** Show that

$$\cos^4 2\theta = A\cos 8\theta + B\cos 4\theta + C$$

where $A$, $B$ and $C$ are rational numbers. *(4 marks)*

**(c)** Hence solve the equation

$$8\cos^4 2\theta = \cos 8\theta + 5$$

for $0 \leqslant \theta \leqslant \pi$, giving each solution in the form $k\pi$. *(3 marks)*

**(d)** Show that

$$\int_0^{\frac{\pi}{2}} \cos^4 2\theta \, \mathrm{d}\theta = \frac{3\pi}{16}$$

*(3 marks)*

January 2013

**8 (a)** Express $-4+4\sqrt{3}\,\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(3 marks)*

**(b) (i)** Solve the equation $z^3=-4+4\sqrt{3}\,\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(4 marks)*

**(ii)** The roots of the equation $z^3=-4+4\sqrt{3}\,\mathrm{i}$ are represented by the points $P$, $Q$ and $R$ on an Argand diagram.

Find the area of the triangle $PQR$, giving your answer in the form $k\sqrt{3}$, where $k$ is an integer. *(3 marks)*

**(c)** By considering the roots of the equation $z^3=-4+4\sqrt{3}\,\mathrm{i}$, show that

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}=0$$ *(4 marks)*

June 2013

**8 (a) (i)** Use de Moivre's theorem to show that

$$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$$

and find a similar expression for $\sin 4\theta$. *(5 marks)*

**(ii)** Deduce that

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$ *(3 marks)*

**(b)** Explain why $t=\tan\frac{\pi}{16}$ is a root of the equation

$$t^4+4t^3-6t^2-4t+1=0$$

and write down the three other roots in trigonometric form. *(4 marks)*

**(c)** Hence show that

$$\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$$ *(5 marks)*

**1 (a)** Express $-9\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

**[2 marks]**

**(b)** Solve the equation $z^4 + 9\mathrm{i} = 0$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

**[5 marks]**

**6 (a) (i)** Use De Moivre's Theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

**[3 marks]**

**(ii)** Write down a similar expression for $z^n + \dfrac{1}{z^n}$.

**[1 mark]**

**(b) (i)** Expand $\left(z - \dfrac{1}{z}\right)^2\left(z + \dfrac{1}{z}\right)^2$ in terms of $z$.

**[1 mark]**

**(ii)** Hence show that

$$8\sin^2\theta\cos^2\theta = A + B\cos 4\theta$$

where $A$ and $B$ are integers.

**[2 marks]**

**(c)** Hence, by means of the substitution $x = 2\sin\theta$, find the exact value of

$$\int_1^2 x^2\sqrt{4 - x^2}\,\mathrm{d}x$$

**[5 marks]**