Core 4: Exponential Growth and Decay

Past Paper Questions
2006 - 2013

Name:

January 2006

4 On 1 January 1900, a sculpture was valued at £80.

When the sculpture was sold on 1 January 1956, its value was £5000.

The value, £V, of the sculpture is modelled by the formula $V = A k^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(a) Write down the value of A. *(1 mark)*

(b) Show that $k \approx 1.07664$. *(3 marks)*

(c) Use this model to:

(i) show that the value of the sculpture on 1 January 2006 will be greater than £200 000; *(2 marks)*

(ii) find the year in which the value of the sculpture will first exceed £800 000. *(3 marks)*

June 2006

8 A disease is spreading through a colony of rabbits. There are 5000 rabbits in the colony. At time t hours, x is the number of rabbits infected. The rate of increase of the number of rabbits infected is proportional to the product of the number of rabbits infected and the number not yet infected.

(a) (i) Formulate a differential equation for $\frac{dx}{dt}$ in terms of the variables x and t and a constant of proportionality k. *(2 marks)*

(ii) Initially, 1000 rabbits are infected and the disease is spreading at a rate of 200 rabbits per hour. Find the value of the constant k.

(You are **not** required to solve your differential equation.) *(2 marks)*

(b) The solution of the differential equation in this model is

$$t = 4 \ln\left(\frac{4x}{5000 - x}\right)$$

(i) Find the time after which 2500 rabbits will be infected, giving your answer in hours to one decimal place. *(2 marks)*

(ii) Find, according to this model, the number of rabbits infected after 30 hours. *(4 marks)*

June 2007

4 A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by

$$x = 15 - 12e^{-\frac{t}{14}}$$

(a) Use this model to find:

(i) the length of a hamster when it is born; *(1 mark)*

(ii) the length of a hamster after 14 days, giving your answer to three significant figures. *(2 marks)*

(b) (i) Show that the time for a hamster to grow to 10 cm in length is given by $t = 14\ln\left(\frac{a}{b}\right)$, where a and b are integers. *(3 marks)*

(ii) Find this time to the nearest day. *(1 mark)*

(c) (i) Show that

$$\frac{dx}{dt} = \frac{1}{14}(15 - x)$$ *(3 marks)*

(ii) Find the rate of growth of the hamster, in cm per day, when its length is 8 cm. *(1 mark)*

January 2008

4 David is researching changes in the selling price of houses. One particular house was sold on 1 January 1885 for £20. Sixty years later, on 1 January 1945, it was sold for £2000. David proposes a model

$$P = Ak^t$$

for the selling price, £P, of this house, where t is the time in years after 1 January 1885 and A and k are constants.

(a) (i) Write down the value of A. *(1 mark)*

(ii) Show that, to six decimal places, $k = 1.079775$. *(2 marks)*

(iii) Use the model, with this value of k, to estimate the selling price of this house on 1 January 2008. Give your answer to the nearest £1000. *(2 marks)*

(b) For another house, which was sold for £15 on 1 January 1885, David proposes the model

$$Q = 15 \times 1.082709^t$$

for the selling price, £Q, of this house t years after 1 January 1885. Calculate the year in which, according to these models, these two houses would have had the same selling price. *(4 marks)*

June 2008

8 (a) The number of fish in a lake is decreasing. After t years, there are x fish in the lake. The rate of decrease of the number of fish is proportional to the number of fish currently in the lake.

(i) Formulate a differential equation, in the variables x and t and a constant of proportionality k, where $k > 0$, to model the rate at which the number of fish in the lake is decreasing. *(2 marks)*

(ii) At a certain time, there were 20 000 fish in the lake and the rate of decrease was 500 fish per year. Find the value of k. *(2 marks)*

(b) The equation

$$P = 2000 - Ae^{-0.05t}$$

is proposed as a model for the number of fish, P, in another lake, where t is the time in years and A is a positive constant.

On 1 January 2008, a biologist estimated that there were 700 fish in this lake.

(i) Taking 1 January 2008 as $t = 0$, find the value of A. *(1 mark)*

(ii) Hence find the year during which, according to this model, the number of fish in this lake will first exceed 1900. *(4 marks)*

January 2009

7 (a) A differential equation is given by $\frac{dx}{dt} = -kte^{\frac{1}{2}x}$, where k is a positive constant.

(i) Solve the differential equation. *(3 marks)*

(ii) Hence, given that $x = 6$ when $t = 0$, show that $x = -2\ln\left(\frac{kt^2}{4} + e^{-3}\right)$. *(3 marks)*

(b) The population of a colony of insects is decreasing according to the model $\frac{dx}{dt} = -kte^{\frac{1}{2}x}$, where x thousands is the number of insects in the colony after time t minutes. Initially, there were 6000 insects in the colony.

Given that $k = 0.004$, find:

(i) the population of the colony after 10 minutes, giving your answer to the nearest hundred; *(2 marks)*

(ii) the time after which there will be no insects left in the colony, giving your answer to the nearest 0.1 of a minute. *(2 marks)*

June 2009

4 A car depreciates in value according to the model

$$V = Ak^t$$

where £V is the value of the car t months from when it was new, and A and k are constants. Its value when new was £12 499 and 36 months later its value was £7000.

(a) (i) Write down the value of A. *(1 mark)*

(ii) Show that the value of k is 0.984 025, correct to six decimal places. *(2 marks)*

(b) The value of this car first dropped below £5000 during the nth month from new. Find the value of n. *(3 marks)*

January 2010

9 A botanist is investigating the rate of growth of a certain species of toadstool. She observes that a particular toadstool of this type has a height of 57 millimetres at a time 12 hours after it begins to grow.

She proposes the model $h = A\left(1 - e^{-\frac{1}{4}t}\right)$, where A is a constant, for the height h millimetres of the toadstool, t hours after it begins to grow.

(a) Use this model to:

(i) find the height of the toadstool when $t = 0$; *(1 mark)*

(ii) show that $A = 60$, correct to two significant figures. *(2 marks)*

(b) Use the model $h = 60\left(1 - e^{-\frac{1}{4}t}\right)$ to:

(i) show that the time T hours for the toadstool to grow to a height of 48 millimetres is given by

$$T = a \ln b$$

where a and b are integers; *(3 marks)*

(ii) show that $\dfrac{dh}{dt} = 15 - \dfrac{h}{4}$; *(3 marks)*

(iii) find the height of the toadstool when it is growing at a rate of 13 millimetres per hour. *(1 mark)*

8 (a) Solve the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{5}(x+1)^{\frac{1}{2}}$$

given that $x = 80$ when $t = 0$. Give your answer in the form $x = \mathrm{f}(t)$. *(6 marks)*

(b) A fungus is spreading on the surface of a wall. The proportion of the wall that is unaffected after time t hours is x%. The rate of change of x is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{5}(x+1)^{\frac{1}{2}}$$

At $t = 0$, the proportion of the wall that is unaffected is 80%. Find the proportion of the wall that will still be unaffected after 60 hours. *(2 marks)*

(c) A biologist proposes an alternative model for the rate at which the fungus is spreading on the wall. The total surface area of the wall is $9\,\mathrm{m}^2$. The surface area that is **affected** at time t hours is $A\,\mathrm{m}^2$. The biologist proposes that the rate of change of A is proportional to the product of the surface area that is affected and the surface area that is unaffected.

(i) Write down a differential equation for this model.

(You are not required to solve your differential equation.) *(2 marks)*

(ii) A solution of the differential equation for this model is given by

$$A = \frac{9}{1 + 4\mathrm{e}^{-0.09t}}$$

Find the time taken for 50% of the area of the wall to be affected. Give your answer in hours to three significant figures. *(4 marks)*

January 2011

5 A model for the radioactive decay of a form of iodine is given by

$$m = m_0 \, 2^{-\frac{1}{8}t}$$

The mass of the iodine after t days is m grams. Its initial mass is m_0 grams.

(a) Use the given model to find the mass that remains after 10 grams of this form of iodine have decayed for 14 days, giving your answer to the nearest gram. *(2 marks)*

(b) A mass of m_0 grams of this form of iodine decays to $\frac{m_0}{16}$ grams in d days.

Find the value of d. *(2 marks)*

(c) After n days, a mass of this form of iodine has decayed to less than 1% of its initial mass.

Find the minimum integer value of n. *(3 marks)*

June 2011

2 The average weekly pay of a footballer at a certain club was £80 on 1 August 1960. By 1 August 1985, this had risen to £2000.

The average weekly pay of a footballer at this club can be modelled by the equation

$$P = Ak^t$$

where £P is the average weekly pay t years after 1 August 1960, and A and k are constants.

(a) (i) Write down the value of A. *(1 mark)*

(ii) Show that the value of k is 1.137411, correct to six decimal places. *(2 marks)*

(b) Use this model to predict the year in which, on 1 August, the average weekly pay of a footballer at this club will first exceed £100 000. *(3 marks)*

4 A scientist is testing models for the growth and decay of colonies of bacteria.

For a particular colony, which is growing, the model is $P = Ae^{\frac{1}{8}t}$, where P is the number of bacteria after a time t minutes and A is a constant.

(a) This growing colony consists initially of 500 bacteria. Calculate the number of bacteria, according to the model, after one hour. Give your answer to the nearest thousand. *(2 marks)*

(b) For a second colony, which is decaying, the model is $Q = 500\,000e^{-\frac{1}{8}t}$, where Q is the number of bacteria after a time t minutes.

Initially, the growing colony has 500 bacteria and, at the same time, the decaying colony has 500 000 bacteria.

(i) Find the time at which the populations of the two colonies will be equal, giving your answer to the nearest 0.1 of a minute. *(3 marks)*

(ii) The population of the growing colony will exceed that of the decaying colony by 45 000 bacteria at time T minutes.

Show that

$$\left(e^{\frac{1}{8}T}\right)^2 - 90e^{\frac{1}{8}T} - 1000 = 0$$

and hence find the value of T, giving your answer to one decimal place. *(4 marks)*

4 The value, £V, of an initial investment, £P, at the end of n years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^n$$

where r% per year is the fixed interest rate.

Mr Brown invests £1000 in Barcelona Bank at a fixed interest rate of 3% per year.

(a) (i) Find the value of Mr Brown's investment at the end of 5 years. Give your value to the nearest £10. *(1 mark)*

(ii) The value of Mr Brown's investment will first exceed £2000 after N complete years.

Find the value of N. *(3 marks)*

(b) Mrs White invests £1500 in Bilbao Bank at a fixed interest rate of 1.5% per year. Mr Brown and Mrs White invest their money at the same time. The value of Mr Brown's investment will first exceed the value of Mrs White's investment after T complete years.

Find the value of T. *(4 marks)*

7 A biologist is investigating the growth of a population of a species of rodent. The biologist proposes the model

$$N = \frac{500}{1 + 9e^{-\frac{t}{8}}}$$

for the number of rodents, N, in the population t weeks after the start of the investigation.

Use this model to answer the following questions.

(a) (i) Find the size of the population at the start of the investigation. *(1 mark)*

(ii) Find the size of the population 24 weeks after the start of the investigation. Give your answer to the nearest whole number. *(1 mark)*

(iii) Find the number of weeks that it will take the population to reach 400. Give your answer in the form $t = r \ln s$, where r and s are integers. *(3 marks)*

(b) (i) Show that the rate of growth, $\frac{dN}{dt}$, is given by

$$\frac{dN}{dt} = \frac{N}{4000}(500 - N)$$ *(4 marks)*

(ii) The maximum rate of growth occurs after T weeks. Find the value of T. *(4 marks)*